

The Scalings of Scalar Structure Functions in a Velocity Field with Coherent Vortical Structures

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In planar turbulence modelled as an isotropic and homogeneous collection of 2-D non-interacting compact vortices, the structure functions $S_p(r)$ of a statistically stationary passive scalar field have the following scaling behaviour in the limit where the Péclet number $Pe \rightarrow \infty$

$$S_p(r) \sim \text{constant} + \ln\left(\frac{r}{LPe^{-1/3}}\right) \text{ for } LPe^{-1/3} \ll r \ll L,$$

$$S_p(r) \sim \left(\frac{r}{LPe^{-1/3}}\right)^{6(1-D)} \text{ for } LPe^{-1/2} \ll r \ll LPe^{-1/3},$$

where L is a large scale and D is the fractal co-dimension of the spiral scalar structures generated by the vortices ($1/2 \leq D < 2/3$). Note that $LPe^{-1/2}$ is the scalar Taylor microscale which stems naturally from our analytical treatment of the advection-diffusion equation. The essential ingredients of our theory are the locality of inter-scale transfer and Lundgren's time average assumption. A phenomenological theory explicitly based only on these two ingredients reproduces our results and a generalisation of this phenomenology to spatially smooth chaotic flows yields $(k \ln k)^{-1}$ generalised power spectra for the advected scalar fields.

I. INTRODUCTION

The theory of turbulent passive scalars has received much attention recently [1,4,7]. The mixing of a scalar field θ in a velocity field \mathbf{v} is governed by the advection-diffusion equation

$$\partial_t \theta(\mathbf{x}, t) + \mathbf{v}(\mathbf{x}, t) \cdot \nabla \theta(\mathbf{x}, t) = \kappa \nabla^2 \theta(\mathbf{x}, t) + f(\mathbf{x}, t) \quad (1.1)$$

where κ denotes the molecular diffusivity of the scalar θ and $f(\mathbf{x}, t)$ is an external source (forcing) driving the scalar. The mixing of a scalar field θ is characterised by its structure functions $S_p \equiv \langle [\theta(\mathbf{x} + \mathbf{r}) - \theta(\mathbf{x})]^p \rangle = \langle (\delta\theta(\mathbf{r}))^p \rangle$ for any number p . If we want to find S_p then we need models of the velocity field \mathbf{v} .

The model which has attracted much attention recently is the Kraichnan model [1] where the velocity field \mathbf{v} is considered to be incompressible, statistically isotropic, white-noise in time (δ -correlated) and Gaussian. Furthermore it has homogeneous increments with power law spatial correlations

$$\begin{aligned} &\langle [v_i(\mathbf{r}, t) - v_i(0, 0)][v_j(\mathbf{r}, t) - v_j(0, 0)] \rangle = \\ &2\delta(t)r^h \left[(h + d - 1)\delta_{ij} - h \frac{r_i r_j}{r^2} \right] \end{aligned} \quad (1.2)$$

where the scaling exponent $h \in [0, 2]$ and d is the dimension of space so that $i, j = 1, 2, \dots, d$. The above tensorial structure of the velocity field is in conformity with incompressibility. The Kraichnan model also assumes a forcing $f(\mathbf{x}, t)$ in (1.1) that is an independent Gaussian random field with zero mean. The forcing is white in time and its covariance is assumed to be a real,

smooth, positive-definite function with rapid decay in space so that the forcing is homogeneous, isotropic and takes place on the integral scale L . The generic scaling behaviour of the structure function $S_p \sim r^{\zeta_p}$, $r \ll L$, of passive scalars in the Kraichnan model was established in [1,4,7,8]. The scaling exponents of this formalism are of the form $\zeta_p = \zeta_p(d, h, p)$ where $h \in [0, 2]$ is the Hölder exponent in (1.2). In the context of this model Balkovsky and Lebedev [2] and Chertkov [3] used the instantonic formalism in a d -dimensional space to find the scaling exponents for large p . It was also shown in the instantonic formalism that $\lim_{p \rightarrow \infty} \zeta_p \simeq \frac{d(2-h)^2}{8h}$ [2] which is independent of p . The scaling exponents were also calculated using other techniques in the limits $h \rightarrow 0$ [4,5], $d \rightarrow \infty$ [6,7] and $p \rightarrow \infty$ [2], and a $2-h$ expansion of ζ_p was proposed in [8]. It was found that ζ_p does depend on p for small values of p in the Kraichnan model.

Our work lies in the opposite extreme of the Kraichnan model. We work in the regime where we have a persistent vortical velocity field frozen in time in two dimensions. The important differences between this model and the Kraichnan model are in the structure of the velocity field infinitely correlated in time in this model but delta correlated in time in the Kraichnan model; and vortical in space in this model, but Gaussian in the Kraichnan model. In this model, the velocity field is that of spatially distributed noninteracting two dimensional vortices with compact structure. We consider the spatial distribution of vortices to be dilute in that they are far from each other and therefore maintain their structure and spatial position for an indefinite period. We also consider this distribution to be homogeneous and isotropic and the velocity field to be incompressible, that is $\nabla \cdot \mathbf{v} = 0$. The model

of the velocity field considered here is an artificial model of planar homogeneous turbulence where the emphasis is on the coherent vortex aspect of the flow. In order of presentation, the first aim of this model is to demonstrate that in the case of the unforced scalar ($f = 0$ in 1.1) we can quantify the statistics of the turbulent scalar field in terms of the scalar's spiral geometry generated by the coherent vortical structures in the flow (sections III and IV). The second aim is to derive the Batchelor k^{-1} power spectrum and all the corresponding structure functions for the scalar field in the case where the scalar is forced and in a statistically steady state (section V). Such a spectrum has been recently observed by Jullien et. al., [9] in a 2-D turbulent flow with well defined, albeit short lived, coherent vortical structures.

In the next section we discuss the scenario of a decaying scalar field in an isolated vortex. In section III, we define structure functions and calculate the spectrum of higher order correlation functions for a single spiral created by a single vortex and decaying by the action of molecular diffusion. In section IV we generalise our analysis to many non-interacting vortices and calculate the scalings of the structure functions of the decaying scalar field. In section V we calculate the generalised power spectra and the corresponding structure functions of statistically stationary scalar spirals by applying the time-average operation approach of Lundgren [10]. In section VI we discuss the phenomenology behind the k^{-1} scalings of the generalised power spectra. Section VII contains conclusions, discussion and the obtainment of the $(k \ln k)^{-1}$ scalings of the generalised power spectra in smooth chaotic flows.

II. PASSIVE SCALAR IN A PLANAR VORTEX

The advection of a decaying passive scalar field by a single planar vortex has been studied by Flohr and Vasilicos [11]. We use the formulation used in [11] namely:

$$\partial_t \theta + \Omega(r) \partial_\phi \theta = \kappa \nabla^2 \theta \quad (2.1)$$

where $\partial_t = \frac{\partial}{\partial t}$ and $\partial_\phi = \frac{\partial}{\partial \phi}$ and $\Omega(r) = \Omega_0 \left(\frac{r}{L}\right)^{-s}$ and L is the maximum distance of the scalar interface from the centre of the vortex. This equation describes the advection and diffusion of a scalar field θ in the azimuthal plane $\mathbf{r} = (r, \phi)$ by a steady vortex with azimuthal velocity component $u_\phi(r) = r\Omega(r) = L\Omega_0 \left(\frac{r}{L}\right)^{1-s}$ and vanishing radial and axial velocity components. Direct Numerical Simulations and experiments in the laboratory have demonstrated the existence and importance of coherent vortices in two-dimensional turbulence and in two-component turbulence in stably-stratified flow with and without rotation of the reference frame [12–15]. Note that axial velocity fields of the form $u_\phi(r) = L\Omega_0 \left(\frac{r}{L}\right)^{1-s}$ have been used in [10, 11, 16–21] and that their large

wavenumber energy spectrum has the form $E(k) \sim k^{-5+2s}$ for $1/2 < s < 2$ with the appropriate large scale bound. We choose $s \geq 1$ to ensure that $u_\phi(r)$ does not increase with increasing r and $s < 2$ to ensure that the energy spectrum is steeper than k^{-1} . The initial scalar field $\theta_0 = \theta(\mathbf{x}, t = 0)$ is characterised by a regular interface between $\theta_0 \neq 0$ and $\theta_0 = 0$ with minimal distance r_0 and maximal L from the rotation axis. By regular structure we mean that the interface has no irregularities on scales smaller than L . Nothing else needs to be specified about the initial scalar field $\theta_0(\mathbf{x})$. Such a patchy initial condition where all the non-zero scalar is confined within a regular interface mimics well initial conditions in certain laboratory experiments where scalar is injected in the flow in the form of blobs (e.g. [9]).

As time proceeds, the patch winds around the vortex and builds up a spiral structure and decays due to diffusion. The characteristic time Ω_0^{-1} is the inverse angular velocity of the vortex at L . This defines a Péclet number $Pe = \Omega_0 L^2 \kappa^{-1}$. We non-dimensionalise equation (2.1) by using the following transformations

$$L^{-1}r \rightarrow r, \Omega_0 t \rightarrow t, \Omega_0^{-1} \Omega(r) \rightarrow \Omega(r), L^2 \nabla^2 \rightarrow \nabla^2,$$

and equation (2.1) takes the form

$$\partial_t \theta + \Omega(r) \partial_\phi \theta = \frac{1}{Pe} \nabla^2 \theta. \quad (2.2)$$

In the non dimensionalised variables we have $\Omega(r) = r^{-s}$, and r_0 represents r_0/L since $L = 1$. Considering finite diffusivity κ , the form of the general solution of the evolution equation (2.2) for any initial field θ_0 in the limit of large t , ie. $t \gg 1$, is the following [11]

$$\begin{aligned} \theta(\mathbf{r}, t) &= \sum_n f_n(r, t) e^{in(\phi - \Omega(r)t)} \\ f_n(r, t) &= f_n(r, 0) e^{[-\frac{1}{3}n^2 \Omega'^2 Pe^{-1} t^3]} \end{aligned} \quad (2.3)$$

where $r = |\mathbf{r}|$ and ϕ is the azimuthal angle and n is an integer. Ω' is the derivative of Ω with respect to r . The angular Fourier coefficients $f_n(r, t)$ are time dependent and the initial condition is fully specified by $f_n(r, 0)$. We do not go in to the details of the solution of (2.1) which can be found in [11].

III. STRUCTURE FUNCTIONS OF PASSIVE DECAYING SCALAR IN ONE VORTEX

In this work we concentrate in finding the scaling properties of the structure functions of the scalar field in a planar turbulence consisting of coherent vortices. The two point equal time p -th order structure function is defined as follows

$$\begin{aligned}\mathcal{S}_p(r, t) &= \overline{\langle [\theta(\mathbf{x} + \mathbf{r}, t) - \theta(\mathbf{x}, t)]^p \rangle} \\ &= \overline{\langle [\delta\theta(\mathbf{r}, t)]^p \rangle}.\end{aligned}\quad (3.1)$$

S_p depends only on the magnitude of the distance between two points, when the ensemble average is taken over an isotropic and homogeneous distribution of the scalar field. The overbar denotes ensemble averaging and the brackets imply space averaging ($\langle \dots \rangle \propto \int d^2\mathbf{x}$).

Let us first calculate $\langle [\delta\theta(\mathbf{r}, t)]^p \rangle$ for one 2-D vortex. We use the binomial expansion as follows

$$\begin{aligned}\langle [\theta(\mathbf{x} + \mathbf{r}, t) - \theta(\mathbf{x}, t)]^p \rangle &= \langle \theta^p(\mathbf{x} + \mathbf{r}, t) \rangle + (-1)^p \langle \theta^p(\mathbf{x}, t) \rangle \\ &+ \sum_{q=1}^{p-1} C_{qp} (-1)^q \langle \theta^{p-q}(\mathbf{x} + \mathbf{r}, t) \theta^q(\mathbf{x}, t) \rangle\end{aligned}\quad (3.2)$$

where C_{qp} is the binomial coefficient of the expansion. In order to calculate $\langle [\delta\theta(\mathbf{r}, t)]^p \rangle$ we first determine

$$\langle \theta^q(\mathbf{x}, t) \theta^{p-q}(\mathbf{x} + \mathbf{r}, t) \rangle = B_{pq}(\mathbf{r}, t) \quad (3.3)$$

which is the q -th term in the binomial expansion of $\langle (\delta\theta(\mathbf{r}))^p \rangle$ as shown in (3.2). Now we write the above as

$$\frac{1}{L_A^2} \int d^2\mathbf{x} \theta^q(\mathbf{x}, t) \theta^{p-q}(\mathbf{x} + \mathbf{r}, t) = B_{pq}(\mathbf{r}, t) \quad (3.4)$$

where L_A is a large scale over which the spatial average may be calculated. The Fourier transform of equation (3.4) is given by

$$\hat{F}_{pq}(\mathbf{k}, t) = \frac{1}{2\pi} \int e^{-i\mathbf{k}\cdot\mathbf{r}} B_{pq}(\mathbf{r}, t) d^2\mathbf{r}. \quad (3.5)$$

Substituting (3.4) in (3.5) and after some standard manipulations we get

$$\begin{aligned}\hat{F}_{pq}(\mathbf{k}, t) &= \frac{1}{2\pi L_A^2} \int \theta^q(\mathbf{x}, t) e^{i\mathbf{k}\cdot\mathbf{x}} d^2\mathbf{x} \\ &\times \int \theta^{p-q}(\mathbf{x}', t) e^{-i\mathbf{k}\cdot\mathbf{x}'} d^2\mathbf{x}'.\end{aligned}\quad (3.6)$$

Now if we integrate (3.6) over a circular shell in k space we get

$$F_{pq}(k, t) = \int_0^{2\pi} dA(k) \hat{F}_{pq}(\mathbf{k}, t) \quad (3.7)$$

where $dA(k) \equiv kd\phi_k$, $k = |\mathbf{k}|$ and ϕ_k is the angle of \mathbf{k} . $F_{pq}(k, t)$ could be called the generalised power spectrum of the scalar field in Fourier space. Substituting (2.3) and (3.6) in (3.7) we get the following

$$\begin{aligned}F_{pq}(k, t) &= \frac{1}{(2\pi L_A^2)} \int dA(k) \\ &\times \int d^2\mathbf{x} e^{i\mathbf{k}\cdot\mathbf{x}} \left\{ \sum_n f_n(x, t) e^{[in(\phi - \Omega(x)t)]} \right\}^q \\ &\times \int d^2\mathbf{x}' e^{-i\mathbf{k}\cdot\mathbf{x}'} \left\{ \sum_m f_m(x', t) e^{[im(\phi' - \Omega(x')t)]} \right\}^{p-q}\end{aligned}\quad (3.8)$$

where $x = |\mathbf{x}|$ and $x' = |\mathbf{x}'|$. After some standard manipulations (3.8) leads to

$$\begin{aligned}F_{pq}(k, t) &= \frac{1}{(2\pi L_A^2)} \int kd\phi_k \int dx x J_n(kx) 2\pi(i)^n e^{in\phi_k} \\ &\times \sum_{n, n_1, n_2, \dots, n_{q-1}} f_{n_1} f_{n_2} \dots f_{n_{q-1}} e^{-in\Omega(x)t} \\ &\times \int dx' x' e^{im\phi_k} J_m^*(kx') 2\pi(-i)^m \\ &\times \sum_{m, m_1, m_2, \dots, m_{p-q-1}} f_{m_1} f_{m_2} \dots f_{m_{p-q-1}} \\ &\times e^{-im\Omega(x')t}\end{aligned}\quad (3.9)$$

where we have changed summation variables in accordance with the following conditions

$$\begin{aligned}n &= n_1 + n_2 + n_3 \dots n_q \\ \text{and } m &= m_1 + m_2 + m_3 + \dots m_{p-q}.\end{aligned}\quad (3.10)$$

All the f_n 's and f_m 's are functions of time t and of x and x' respectively. $J_n(kx)$ is the Bessel function which has been substituted in place of the integral representation

$$\int_0^{2\pi} e^{in\phi} e^{-ikx\cos(\phi - \phi_k)} d\phi = 2\pi(i)^n e^{in\phi_k} J_n(kx). \quad (3.11)$$

After integrating (3.9) with respect to ϕ_k and summing over m we get the following relation

$$\begin{aligned}F_{pq}(k, t) &= \frac{1}{(2\pi L_A^2)} \int dx k x J_n(kx) 2\pi(i)^n \\ &\times \sum_{n, n_1, n_2, \dots, n_{q-1}} f_{n_1} f_{n_2} \dots f_{n_{q-1}} e^{-in\Omega(x)t} \\ &\times \int dx' x' J_{-n}^*(kx') 2\pi(-i)^{-n} \\ &\times \sum_{m_1, m_2, \dots, m_{p-q-1}} f_{m_1} f_{m_2} \dots f_{m_{p-q-1}} \\ &\times e^{in\Omega(x')t}.\end{aligned}\quad (3.12)$$

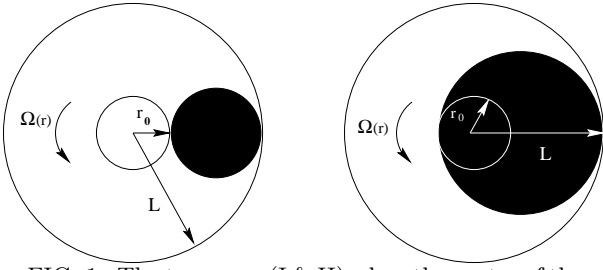


FIG. 1. The two cases (I & II) when the centre of the vortex is respectively outside a scalar patch and inside it.

We now have two cases to study (see figure 1). In case I the vortex centre is outside the scalar patch and its nearest distance to the scalar interface is r_0 . In Case II the centre of the vortex is inside the scalar patch and its nearest distance to the scalar interface is r_0 . In view of the above we can write (3.12) as

$$\begin{aligned}
F_{pq}(k, t) &= \frac{1}{(2\pi L_A^2)} \left[\int_0^{r_0} + \int_{r_0}^\infty \right] dx k x J_n(kx) 2\pi(i)^n \\
&\times \sum_{n, n_1, n_2 \dots n_{q-1}} f_{n_1} f_{n_2} \dots f_{n_{q-1}} e^{-in\Omega(x)t} \\
&\times \left[\int_0^{r_0} + \int_{r_0}^\infty \right] dx' x' J_{-n}^*(kx') 2\pi(-i)^{-n} \\
&\times \sum_{m_1, m_2 \dots m_{p-q-1}} f_{m_1} f_{m_2} \dots f_{m_{p-q-1}} \\
&\times e^{in\Omega(x')t}.
\end{aligned} \tag{3.13}$$

In (3.13) we have four terms of the form as shown below

$$\begin{aligned}
&\int_0^{r_0} dx \int_0^{r_0} dx' (\dots) + \int_0^{r_0} dx \int_{r_0}^\infty dx' (\dots) \\
&+ \int_{r_0}^\infty dx \int_0^{r_0} dx' (\dots) + \int_{r_0}^\infty dx \int_{r_0}^\infty dx' (\dots)
\end{aligned} \tag{3.14}$$

with the integrands denoted by (\dots) being the same as (3.13) for all the terms of the above. For case I, terms containing $\int_0^{r_0} (\dots)$ are zero since $f_n = 0$ in the regime $0 < x < r_0$ for all n , even for $n = 0$, since there is no scalar patch in the region $0 < x < r_0$. Therefore contributions only come from the term

$$\int_{r_0}^\infty dx x \int_{r_0}^\infty dx' x' (\dots).$$

For case II we can legitimately replace θ by $\theta - f_0$ and get the same result as in case I. Hence the only contributing term is the following

$$\begin{aligned}
F_{pq}(k, t) &= \frac{1}{(2\pi L_A^2)} \int_{r_0}^\infty dx k x J_n(kx) 2\pi(i)^n \\
&\times \sum_{n, n_1, n_2 \dots n_{q-1}} f_{n_1} f_{n_2} \dots f_{n_{q-1}} e^{-in\Omega(x)t}
\end{aligned}$$

$$\begin{aligned}
&\times \sum_{n, n_1, n_2 \dots n_{q-1}} f_{n_1} f_{n_2} \dots f_{n_{q-1}} e^{-in\Omega(x)t} \\
&\times \int_{r_0}^\infty dx' x' J_{-n}^*(kx') 2\pi(-i)^{-n} \\
&\times \sum_{m_1, m_2 \dots m_{p-q-1}} f_{m_1} f_{m_2} \dots f_{m_{p-q-1}} \\
&\times e^{in\Omega(x')t}.
\end{aligned} \tag{3.15}$$

It is because f_0 decays with a time scale which is much larger than the decay time of the non-zero harmonics [16] that f_0 is considered to be a constant and therefore subtracted away from the θ in case II. The same reasoning can be applied for case I. Hence all the n_i 's and m_i 's are non-zero in the above equation and in the rest of the paper.

To take into account the fact that diffusion gradually wipes out the spiral structure of the scalar field near the vortex centre (a fact not taken into account in [16] where the spiral structure is assumed to exist wholly intact until finally destroyed by viscosity), we follow Flohr and Vassilicos [11] and define a critical radius ρ which gives a measure of this diffused region. This critical radius is defined in the limit $Pe \rightarrow \infty$ for times $t \ll Pe^{1/3}$ which are such that

$$f_n(r, t) = f_n(r, 0) \text{ for } r \ll \rho$$

but

$$|f_n(r, t)| \ll |f_n(r, 0)| \text{ for } r \gg \rho \text{ see (2.3).}$$

Hence, we set $\frac{1}{3}n^2\Omega'^2(\rho)Pe^{-1}t^3 = 1$ which implies

$$\rho(t) = \left[\frac{1}{3}n^2s^2Pe^{-1}t^3 \right]^{\frac{1}{2(s+1)}}.$$

This critical radius is time-dependent and grows with time. It can be thought of as a diffusive length scale over which the harmonics in n have diffused and the spiral structure has been smeared out.

In view of the above, the integrals in (3.15) can be further divided as

$$\int_{r_0}^\infty dx = \int_{r_0}^\rho dx + \int_\rho^\infty dx.$$

The only significantly non-zero contribution comes from the range $\rho < x < 1$ in the integrals and similarly for x' . Hence we get the following

$$\begin{aligned}
F_{pq}(k, t) &= \frac{1}{(2\pi L_A^2)} \int_\rho^\infty dx k x J_n(kx) 2\pi(i)^n \\
&\times \sum_{n, n_1, n_2 \dots n_{q-1}} f_{n_1} f_{n_2} \dots f_{n_{q-1}} e^{-in\Omega(x)t}
\end{aligned}$$

$$\begin{aligned}
& \times \int_{\rho}^{\infty} dx' x' J_{-n}^*(kx') 2\pi(-i)^{-n} \\
& \times \sum_{m_1, m_2 \dots m_{p-q-1}} f_{m_1} f_{m_2} \dots f_{-n-m_1 \dots -m_{p-q-1}} \\
& \times e^{in\Omega(x')t}
\end{aligned} \tag{3.16}$$

Now for large kx , ie. $kx \gg 1$, we can use the asymptotic expansion for the Bessel function

$$J_n(kx) \sim \left(\frac{1}{2\pi kx}\right)^{\frac{1}{2}} \left[(-i)^{n+\frac{1}{2}} e^{ikx} + (i)^{n+\frac{1}{2}} e^{-ikx}\right]. \tag{3.17}$$

This is appropriate for our analysis if the Fourier modes are to resolve at the very least the distance r_0 from the centre of the vortex to the scalar patch interface i.e $1 < kr_0$. After substituting (3.17) in (3.16) we use the method of stationary phase to evaluate the integrals where the phase is given by

$$\Phi = kx - n\Omega(x)t. \tag{3.18}$$

The approximation for a general integral of this type is known to be

$$\begin{aligned}
I(x) &= \int_a^b f(t) \exp[ix\Psi(t)] dt \\
&\sim \sqrt{\frac{\pi}{2x |\Psi''(t^*)|}} f(t^*) \exp[ix\Psi(t^*) \pm \pi/4]
\end{aligned} \tag{3.19}$$

where t^* is the t where the derivative of the phase is zero. The condition of stationary phase gives

$$0 = \Phi' = -k - n\Omega'(x_n)t \tag{3.20}$$

which picks out points x_n where the contribution to the integral is maximum. Finally what we get is

$$\begin{aligned}
F_{pq}(k, t) &\sim \\
&\sum_{n, n_1 \dots n_{q-1}} \frac{2\pi k}{n |\Omega''(x_n)| t} \\
&\times \left(\frac{x_n}{2\pi k}\right) f_{n_1} f_{n_2} \dots f_{n-n_1-n_2-\dots-n_{q-1}} \\
&\times \sum_{m_1, m_2 \dots m_{p-q-1}} f_{m_1} f_{m_2} \dots f_{-n-m_1-m_2-\dots-m_{p-q-1}}
\end{aligned} \tag{3.21}$$

where $f_n = f_n(x_n, 0)$ and similarly for f_m . Now from the condition of stationary phase (3.20) we can find

$$x_n = \left(\frac{sn t}{k}\right)^{\frac{1}{s+1}} \tag{3.22}$$

where we have used $\Omega(r) = r^{-s}$. The stationary phase contributes only when $\rho < x_n < 1$ because the spiral

structure only exists in that range of distances from the centre of the vortex. The relation between the fractal co-dimension (Kolmogorov capacity) D of the scalar spiral and the power law of the decay of the azimuthal velocity of the vortex is [11], [22]

$$D = \frac{s}{s+1}. \tag{3.23}$$

This D is such that $1/2 < D < 2/3$ because $1 < s < 2$ and gives a measure of the space-filling property of the spiral. Hence after doing the summations in (3.21) we can show that the power spectrum $F_{pq}(k, t)$ scales like

$$\begin{aligned}
F_{pq}(k, t) \\
\sim k^{-(3-2D)} t^{2(1-D)} [\text{const} + \text{higher order terms}]
\end{aligned} \tag{3.24}$$

in the limit $Pe \rightarrow \infty$ and in the range $t < k < \sqrt{\frac{Pe}{t}}$ for times $1 \ll t \ll Pe^{1/3}$ which is the range of times for which the scalar patch has a well-defined spiral structure in the range of wavenumbers $t < k < \sqrt{\frac{Pe}{t}}$ (obtained from $\rho < x_n < 1$). The higher order terms are functions of k/t , and decay faster than $(k/t)^{-1}$ in the range $t < k < \sqrt{\frac{Pe}{t}}$, and can therefore be neglected.

We notice that as time runs forward the spiral range ($t < k < \sqrt{\frac{Pe}{t}}$) shifts to higher values of k which is solely due to the vortex continuously wrapping the scalar field in to finer and finer spirals thus generating scales which have higher wavenumbers. This range also shrinks as it shifts to higher values of k because of the action of diffusion. In the next section we generalise our results to the case of multiple spirals generated by a dilute collection of noninteracting vortices which may be representative of a turbulent velocity field with coherent vortical structure, perhaps obtained in the experiments of Jullien et al. [9].

IV. STRUCTURE FUNCTIONS OF PASSIVE DECAYING SCALAR IN A FLOW CONSISTING OF MANY IDENTICAL NON-INTERACTING VORTICES

All the analysis in this section is carried out in dimensionalised variables so we invert the transformations of section II. Let us consider many non interacting vortices randomly distributed over 2-D space and sufficiently far apart so that we can safely describe the velocity field in terms of compact vortical structures characterised by

$$\Omega(x) = \Omega_0 \left(\frac{x}{L}\right)^{-s} \text{ if } \frac{x}{L} \leq 1$$

$$\Omega(x) = 0 \text{ if } \frac{x}{L} > 1$$

where the x 's are measured from the centre of each vortex at \mathbf{x}_m for all m and

$$\min|\mathbf{x}_m - \mathbf{x}_n| \gg L; \text{ for all } m \text{ and } n. \quad (4.1)$$

For the calculation of the generalised power spectrum and structure functions we need only to consider the scalar patches within a distance L of each vortex because these scalar patches acquire a spiral structure and thereby dominate the scaling of the structure functions. The scalar field at distances larger than L from all vortices contributes an $\mathcal{O}(r/L)$ term to the structure function for $r \ll L$ because the interfacial structure of the scalar field far from the vortices remain regular. As we show in this section, this term is negligible in the $r/L \ll 1$ limit compared to the r -dependence of the structure functions caused by the scalar spiral structures around the vortices. It is therefore sufficient to consider that $\theta(\mathbf{x}, t)$ consists only of the local scalar fields $\theta_m(\mathbf{x} - \mathbf{x}_m, t)$ in the vicinity of vortices labelled m and write

$$\theta(\mathbf{x}, t) = \sum_m \theta_m(\mathbf{x} - \mathbf{x}_m, t). \quad (4.2)$$

Every scalar spiral in the right hand side of (4.2) is localised within a distance L of \mathbf{x}_m and the condition (4.1) ensures that they do not overlap each other.

Now we can generalise (3.6) to include the effect of many non-interacting vortices with non-overlapping scalar spirals as shown below

$$\begin{aligned} \hat{F}_{pq}(\mathbf{k}, t) &= \frac{1}{2\pi L_A^2} \sum_m \int \theta_m^q(\mathbf{x}, t) e^{i\mathbf{k} \cdot \mathbf{x}} d^2\mathbf{x} \\ &\times \int \theta_m^{p-q}(\mathbf{x}', t) e^{-i\mathbf{k} \cdot \mathbf{x}'} d^2\mathbf{x}'. \end{aligned} \quad (4.3)$$

Since the integrals are independent of the m 's we can take them out of the sum. Hence we get the same result as in (3.24) multiplied by the number of vortices per unit area, that is

$$F_{pq}(k, t) \sim (kL)^{-(3-2D)} (\Omega_0 t)^{2(1-D)} \sum_m \frac{1}{2\pi L_A^2}. \quad (4.4)$$

This asymptotic relation is valid when $\Omega_0 t < kL < \sqrt{\frac{Pe}{\Omega_0 t}}$ which is found from the condition $\rho < x_n < l$ in dimensionalised form and $1 \ll \Omega_0 t \ll Pe^{1/3}$ in the limit $Pe \rightarrow \infty$.

Assuming the distribution of the scalar spirals over the two dimensional space to be homogeneous and isotropic, the power spectrum $F_{pq}(k, t) = 2\pi k \overline{\hat{F}_{pq}(\mathbf{k}, t)}$ where the bar implies ensemble averaging over the distribution of many spirals (because vortices are non-interacting and spirals are therefore statistically independent from each other, it does make sense for the average over space already included in the definition of $\hat{F}_{pq}(\mathbf{k}, t)$ to be taken over an idealised space where there is only one spiral, and

for the ensemble average to be taken over the distribution of many spirals). From (3.2) we can show that for a homogenous distribution of scalar spirals the odd order structure functions vanish. Only the even order structure functions do not vanish, that is for $p = \text{even}$. Hence from (3.1)

$$\begin{aligned} \mathcal{S}_p(r, t) &= \overline{[\theta(\mathbf{x} + \mathbf{r}, t) - \theta(\mathbf{x}, t)]^p} \\ &= \overline{[\delta\theta(\mathbf{r}, t)]^p} \\ &\sim \int \left(2 - \sum_{q=1}^{p-1} C_{qp} e^{i\mathbf{k} \cdot \mathbf{r}} \right) \overline{\hat{F}_{pq}(\mathbf{k}, t)} k dk d\phi \\ &\sim \int (1 - e^{i\mathbf{k} \cdot \mathbf{r}}) F_{pq}(k, t) k dk d\phi \\ &\sim \int (1 - J_0(kr)) F_{pq}(k, t) dk \end{aligned} \quad (4.5)$$

where $J_0(kr)$ is same as (3.11) with $n = 0$. After substituting (4.4) in (4.5) and integrating we find

$$\begin{aligned} \mathcal{S}_p(r, t) &\sim \left(\frac{r}{L} \right)^{2(1-D)} (\Omega_0 t)^{2(1-D)} \\ &\text{in the ranges } \frac{1}{\Omega_0 t} > \frac{r}{L} > \sqrt{\frac{\Omega_0 t}{Pe}} \\ &\text{and } 1 \ll \Omega_0 t \ll Pe^{1/3}. \end{aligned} \quad (4.6)$$

V. STRUCTURE FUNCTIONS OF STATISTICALLY STATIONARY PASSIVE SCALAR

To achieve a statistically stationary passive scalar field we may imagine that, as scalar patches take spiral shapes and decay, more scalar patches are introduced in to the flow as may well happen in an experimental setup in the laboratory. This procedure soon leads to a situation where many scalar spirals coexist in the flow all in different stages of their evolution. Assuming the rate of regular injection of the scalar blobs to balance exactly the rate of scalar dissipation, we can expect to have a statistically stationary scalar field. In this case, the averaging over many spirals in different stages of their evolution (which is involved in the calculation of the generalised power spectra and structure functions) may be assumed, in the spirit of Lundgren [10], to be equivalent to averaging over the life-time of a single spiral. Hence, to obtain the generalised power spectra of the statistically stationary scalar we average the previous section's results over time in the range $1 < \Omega_0 t < Pe^{1/3}$, which represents the life time of the spirals. The spiral structure lies in the range $\rho < x_n < L$ which implies $\Omega_0 t < kL < \sqrt{\frac{Pe}{\Omega_0 t}}$. This spiral range of wavenumbers together with the time range of the spiral gives the range of $\Omega_0 t$ over which we

can average for a given value of kL . This leads to a time averaged $F_{pq}(k, t)$ which takes the form

$$F_{pq}(k) \sim (kL)^{-1} \quad \text{where} \quad 1 < kL < Pe^{1/3}, \quad (5.1)$$

$$F_{pq}(k) \sim (kL)^{-(7-6D)} Pe^{2(1-D)} \quad \text{where} \quad Pe^{1/3} < kL < Pe^{1/2}. \quad (5.2)$$

(5.1) and (5.2) are the result of averaging (4.4) over the time ranges $1 < \Omega_0 t < kL$ and $1 < \Omega_0 t < \frac{Pe}{(kL)^2}$ respectively. These time ranges are determined by the respective wavenumber ranges in (5.1) and (5.2). Finally (5.1) and (5.2) give the following structure functions

$$S_p(r) \sim \text{constant} + \ln\left(\frac{r}{LPe^{-1/3}}\right) \quad \text{where} \quad LPe^{-1/3} < r < L, \quad (5.3)$$

$$S_p(r) \sim \left(\frac{r}{LPe^{-1/3}}\right)^{6(1-D)} \quad \text{where} \quad LPe^{-1/2} < r < LPe^{-1/3}. \quad (5.4)$$

In (5.3) and (5.4) $S_p(r)$ is a time average of $S_p(r, t)$ in (4.6). Note that $\zeta_p = 0$ with a logarithmic correction in the range $LPe^{-1/3} < r < L$ and that $\zeta_p = 6(1-D) \in]2, 3[$ in the dissipative range $LPe^{-1/2} < r < LPe^{-1/3}$.

The structure functions $S_p(r)$ are not time-dependent and may be interpreted as characterising a scalar field in a statistically steady state achieved with an external large-scale source of scalar (scalar forcing). This scalar forcing may consist of regularly placing in the flow scalar blobs with large-scale smooth interfacial structure. In the spirit of Birkhoff's Ergodic theorem [23] we should expect Lundgren's time average assumption to be relevant for the calculation of structure functions when the scalar field is statistically stationary.

VI. PHENOMENOLOGY

In this section we extract the phenomenology underlying the calculations and results of the previous sections and show that the essential ingredient of this phenomenology are the locality of scalar inter-scale transfer (6.1) and the Lundgren time-averaging operation. Indeed, as we show in this section, (5.1) and (5.2) can be retrieved by a simple phenomenological argument based on these two ingredients.

Let us return to the time-dependent wind-up of scalar spirals. As time proceeds, ie. as $\Omega_0 t \rightarrow \Omega_0(t + \delta t)$, then $kL \rightarrow kL + L\delta k$ because of the differential rotation (which amounts to local shear) in every steady vortex and the entire scalar spectrum is shifted towards higher wavenumbers with time (see Figure 2). That is to say that the

shearing advection to which the scalar patches are subjected in every steady vortex is such that the generalised power spectra obey

$$F_{pq}(kL + \delta kL, \Omega_0 t + \Omega_0 \delta t) d(kL + \delta kL) = F_{pq}(kL, \Omega_0 t) d(kL) \quad (6.1)$$

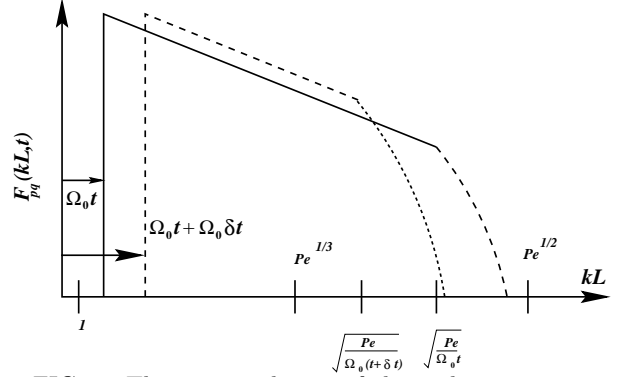


FIG. 2. The time evolution of the scalar power spectrum $F_{pq}(kL, t)$ in a log-log plot.

The amount of scalar variance in the wavenumber band $d(kL)$ around wavenumber kL is simply transported to wavenumber band $d(kL + \delta kL)$ around wavenumber $kL + \delta kL$ after an incremental time duration $\Omega_0 \delta t$ (see Figure 2). As shown in [20], the distance l between consecutive coils of the scalar spiral in every vortex at a distance r from the centre scales as

$$l \sim \frac{L}{\Omega_0 t} \left(\frac{r}{L}\right)^{1+1/s}. \quad (6.2)$$

Letting time vary by a small amount δt , the distance between two coils changes by

$$\frac{\delta l}{l} \simeq -\frac{\delta t}{t}$$

the minus sign indicating that l is decreasing. Identifying k with $\frac{2\pi}{l}$ for the purpose of equation (6.1) so that $\frac{\delta k}{k} = -\frac{\delta l}{l}$, it follows that (6.1) becomes

$$F_{pq} \left[kL \left(1 + \frac{\delta t}{t} \right), \Omega_0 t \left(1 + \frac{\delta t}{t} \right) \right] dkL \left(1 + \frac{\delta t}{t} \right) = F_{pq}(kL, \Omega_0 t) d(kL). \quad (6.3)$$

The solution of this equation is

$$F_{pq}(kL, \Omega_0 t) = L(kL)^{-1} \mathcal{F}_{pq} \left(\frac{\Omega_0 t}{kL} \right) \quad (6.4)$$

where \mathcal{F}_{pq} are arbitrary dimensionless functions. As indicated in figure 2 this form of the generalised spectra is valid in the limit $Pe \rightarrow \infty$ and in the wavenumber range $\Omega_0 t \ll kL \ll \sqrt{\frac{Pe}{\Omega_0 t}}$ and time range $1 \ll \Omega_0 t \ll Pe^{1/3}$.

Note that $1 \ll \Omega_0 t \ll Pe^{1/3} \ll \sqrt{\frac{Pe}{\Omega_0 t}} \ll Pe^{1/2}$. The inverse of $\Omega_0 t$ represents the decaying outer length-scale of the spiral range and the inverse of $\sqrt{\frac{Pe}{\Omega_0 t}}$ represents the growing micro-scale of diffusive attrition. A wavenumber in the range $1 \ll kL \ll Pe^{1/3}$ during the time-period $1 \ll \Omega_0 t \ll Pe^{1/3}$ does not have the time to be affected by diffusive attrition and only receives scalar variance activity from lower wavenumbers until $\Omega_0 t = kL$. We therefore refer to $1 \ll kL \ll Pe^{1/3}$ as the advective wavenumber range. The time averaged generalised power spectra in this range are given by

$$F_{pq}(kL) = \frac{1}{kL-1} \int_1^{kL} d(\Omega_0 t) L(kL)^{-1} \mathcal{F}_{pq} \left(\frac{\Omega_0 t}{kL} \right) \quad (6.5)$$

and \mathcal{F}_{pq} must be increasing functions of $\frac{\Omega_0 t}{kL}$ because the differential rotation's shearing process causes the power spectra to shift from small to large wavenumbers (see figure 2). Hence we retrieve (5.1), i.e.

$$F_{pq}(kL) \sim (kL)^{-1}$$

in the advective range $1 \ll kL \ll Pe^{1/3}$ which is well defined in the limit $Pe \rightarrow \infty$.

In the advective-diffusive range $Pe^{1/3} \ll kL \ll Pe^{1/2}$ a wavenumber kL experiences the advection process from $\Omega_0 t = 1$ until $kL = \sqrt{\frac{Pe}{\Omega_0 t}}$ when molecular diffusion sets in. Hence the time averaged generalised power spectra are given by

$$F_{pq}(kL) = \frac{1}{\frac{Pe}{kL^2} - 1} \int_1^{\frac{Pe}{kL^2}} d(\Omega_0 t) L(kL)^{-1} \mathcal{F}_{pq} \left(\frac{\Omega_0 t}{kL} \right) \quad (6.6)$$

in the advective diffusive range and using $\mathcal{F}_{pq} \left(\frac{\Omega_0 t}{kL} \right) \sim \left(\frac{\Omega_0 t}{kL} \right)^{2(1-D)}$ (see (4.4)) we retrieve (5.2), i.e.

$$F_{pq}(kL) \sim Pe^{2(1-D)} (kL)^{-7+6D}$$

in the advective-diffusive range $Pe^{1/3} \ll kL \ll Pe^{1/2}$ which is well defined in the limit $Pe \rightarrow \infty$. Note that the diffusive micro-length-scale $LPe^{-1/2}$ is the Taylor microscale of the scalar field (first introduced by Corrsin in 1951). Note also that $7-6D \in]3, 4[$ and that the experimental results of [9] seem to show a steeper power-law wavenumber spectrum at wavenumbers larger than where the k^{-1} spectrum is observed.

VII. CONCLUSIONS AND DISCUSSION

In a two-dimensional isotropic and homogeneous collection of non-interacting compact and time-independent

singular vortices with a large-wavenumber energy spectrum $E(k) \sim k^{-\alpha}$ with $1 < \alpha \leq 3$, the structure functions of an advected and freely decaying scalar field have the following scaling behaviour in the limit where $Pe \rightarrow \infty$

$$\mathcal{S}_p(r, t) \sim \left(\frac{r}{L} \Omega_0 t \right)^{2(1-D)} \quad (7.1)$$

where $\sqrt{\frac{\Omega_0 t}{Pe}} \ll \frac{r}{L} \ll \frac{1}{\Omega_0 t}$ and $1 \ll \Omega_0 t \ll Pe^{1/3}$, and the fractal co-dimension D of the scalar interfaces is such that $1/2 \leq D < 2/3$.

By applying the Lundgren time-average assumption we obtain predictions for the structure functions of a statistically stationary scalar field in the same 2-D velocity field and the same limit $Pe \rightarrow \infty$

$$\mathcal{S}_p(r) \sim \text{constant} + \ln \left(\frac{r}{LPe^{-1/3}} \right) \quad (7.2)$$

in the range $LPe^{-1/3} \ll r \ll L$ and

$$\mathcal{S}_p(r) \sim \left(\frac{r}{LPe^{-1/3}} \right)^{6(1-D)} \quad (7.3)$$

in the range $LPe^{-1/2} \ll r \ll LPe^{-1/3}$. The logarithmic term in (7.2) corresponds to k^{-1} generalised power spectra. It may be worth mentioning that the 2-D velocity fields of Holzer and Siggia [24] where they observe a well-defined k^{-1} scalar power spectrum are replete with spiral scalar structures.

Predictions of k^{-1} scalar power spectra in the limit $Pe \rightarrow \infty$ have been made for scalar fields in smooth (ie. at least everywhere continuous and differentiable) homogeneous and isotropic random velocity field with arbitrary dimensionality and time dependence by Chertkov et. al. [25] who generalised and unified the results of Batchelor [26] and Kraichnan [27]. Experimental investigations of the high Péclet number k^{-1} scalar power spectrum have been inconclusive in 3-D turbulent flows even though Prasad and Sreenivasan have claimed such a spectrum in a 3-D wake [28]. However k^{-1} scalar power spectra have been observed at high Péclet numbers in numerical simulations of scalar fields in 2-D and 3-D chaotic flows [29–31] and in 2-D velocity fields obeying the stochastically forced Euler equation restricted to a narrow band of small (integral scale) wavenumbers [24]. More recently, k^{-1} scalar power spectra have been observed at $Pe = 10^7$ in 2-D or quasi 2-D statistically stationary turbulent flows in the same range where the velocity field's energy spectrum is k^{-3} by Jullien et al. [9] who have also observed logarithmic scalings in that range for all order structure functions (similarly to (7.2), but without the ability to establish the $LPe^{-1/3}$ scaling factor and range). The theory of Chertkov et al. [25] does not apply to this experiment because homogeneous and isotropic random velocity fields which are everywhere continuous and differentiable have energy spectra steeper than k^{-4} (see Appendix). The present paper's theory,

however, applies when the energy spectrum scales as $k^{-\alpha}$ with $1 < \alpha \leq 3$ but is limited to time-independent velocity fields. Nevertheless the phenomenology developed in section VI also holds for time-dependent velocity fields and we now apply and generalise it to spatially smooth chaotic flows (and also, by the way, to frozen straining velocity field structures).

The starting point of our phenomenology is the locality of transfer (6.1). Pedrizzetti and Vassilicos [21] have shown that inter-scale transfer in 2-D compact vortices is indeed local at a given scale when velocity gradients do not vary much in physical space over that scale. This is the case in the 2-D axisymmetric vortices considered in this paper but also in spatially smooth velocity fields. In a spatially smooth chaotic flow the distance l between successive folds of the scalar interface decays exponentially as determined by the largest positive Lyapunov exponent λ , ie. $l(t) \sim e^{-\lambda t}$ which implies $\frac{dl}{l} \simeq -\lambda dt$. Applying the locality of transfer property we get

$$\begin{aligned} F_{pq}(k(1 + \lambda \delta t), t + \delta t) dk(1 + \lambda \delta t) \\ = F_{pq}(k, t) dk \end{aligned} \quad (7.4)$$

the solution of which is

$$F_{pq}(k, t) = k^{-1} \mathcal{F}_{pq} \left(\frac{e^{\lambda t}}{k} \right). \quad (7.5)$$

This form of the generalised spectra is valid for $Pe \rightarrow \infty$ and as long as $1 < e^{\lambda t} < k$, so that applying Lundgren's time-average operation from $t = 0$ to $t = \lambda^{-1} \ln k$ gives

$$\begin{aligned} F_{pq}(k) &= k^{-1} \frac{\lambda}{\ln k} \int_0^{\frac{\ln k}{\lambda}} \mathcal{F}_{pq} \left(\frac{e^{\lambda t}}{k} \right) dt \\ &\sim [k \ln k]^{-1} \end{aligned} \quad (7.6)$$

because \mathcal{F}_{pq} is an increasing function of $\frac{e^{\lambda t}}{k}$. Power spectra of scalar fields in spatially smooth chaotic flows are believed to scale as k^{-1} in the limit $Pe \rightarrow \infty$ [29–31] but our theory predicts $(k \ln k)^{-1}$. This is a small correction to the spectrum but an exponentially large correction to the scalar variance in the limit $Pe \rightarrow \infty$.

VIII. ACKNOWLEDGMENTS

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IX. APPENDIX

For a statistically homogeneous velocity field with velocity components $u_i(\mathbf{x})$ we can define a correlation function

$$R_{ij}(\mathbf{r}) = \overline{\langle u_i(\mathbf{x}) u_j(\mathbf{x} + \mathbf{r}) \rangle}$$

and its Fourier transform

$$\Phi_{ij}(\mathbf{k}) = \frac{1}{(2\pi)^2} \int d\mathbf{r} R_{ij}(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}}.$$

Incompressibility ($\nabla \cdot \mathbf{u} = 0$) and statistical isotropy of a two-component or 2-D velocity field $u_i(\mathbf{x})$, $i = 1, 2$, imply

$$\Phi_{ij}(\mathbf{k}) = \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \frac{E(k)}{\pi k}$$

where the average kinetic energy per unit mass of the velocity field is $E = \int_0^\infty dk E(k)$, $k \equiv |\mathbf{k}|$; $E(k)$ is the energy spectrum of the velocity field.

One dimensional energy spectra are defined as follows

$$\phi_{ij}(k_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{ij}(r_1, 0) e^{-ik_1 r_1} dr_1$$

which, because of isotropy, are completely characterised by a single component, say $\phi_{11}(k_1)$.

From the above and a few standard manipulations one can get

$$\int_1^\infty dx \frac{\sqrt{x^2 - 1}}{\pi x^2} \frac{E(x k_1)}{\phi_{11}(k_1)} = \frac{1}{2}$$

and in a range of wavenumbers where both $E(k)$ and $\phi_{11}(k_1)$ are monotonically decreasing functions of k and k_1 respectively, it follows that $E(k) \sim k^{-p}$ if and only if $\phi_{11}(k_1) \sim k_1^{-p}$.

The pivotal assumption in [25] is that the velocity field $u_i(\mathbf{x})$ is Taylor-expandable up to at least first derivative terms everywhere in physical space. In particular, this means that $u_1(x_1, 0)$ is differentiable with respect to x_1 everywhere on the x_1 axis. If the first derivative of $u_1(x_1, 0)$ with respect to x_1 is also continuous everywhere along the x_1 axis then the Fourier transform $\hat{u}_1(k_1)$ of $u_1(x_1, 0)$ must decay faster than k_1^{-2} [32]. If, however, the first derivative of $u_1(x_1, 0)$ is not everywhere continuous, then it is discontinuous either on a set of well-separated points or on a more pathological set of points which accumulate (and are therefore not well-separated) in a fractal-like or in a spiral-like manner [22, 32]. In the case where discontinuities in the derivative field of $u_1(x_1, 0)$ are well-separated, $\hat{u}_1(k_1)$ decays as k_1^{-2} because the Fourier transform of well-separated discontinuities between which the field is continuous decays as k_1^{-1} and the Fourier transform of the derivative of $u_1(x_1, 0)$ is equal to $ik_1 \hat{u}_1(k_1)$. In the other case where discontinuities are not well-separated and accumulate, the decay of $\hat{u}_1(k_1)$ can be anywhere between k_1^{-1} and k_1^{-2} [32, 33],

but in this case $u_1(x_1, 0)$ cannot be considered to be differentiable at those points where discontinuities of its derivative accumulate.

In conclusion, the differentiability of the velocity field everywhere in physical space implies that $\hat{u}_1(k_1)$ must decay at least as k_1^{-2} and therefore $\phi_{11}(k_1) \sim |\hat{u}_1(k_1)|^2 \sim \mathcal{O}(k_1^{-4})$ which in turn implies $E(k) \sim \mathcal{O}(k^{-4})$.

The condition $E(k) \sim \mathcal{O}(k^{-3})$ stated in the conclusion of [25] guarantees that the strain field is large-scale but not that the velocity field is differentiable. The spectral condition required to use the pivotal assumption of differentiability in [25] should in fact be $E(k) \sim \mathcal{O}(k^{-4})$.

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